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On the Vacuum Theta Angle in Yang-Mills Field Theories

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Abstract

Recently, MIT group found a numerical solution to the non-abelian gauge field theory. This solution is shown to contain non-integer topological numbers. By using this fact, V.V.Khoze has proved that the vacuum theta-angle is zero in non-abelian gauge field theories. Here, we study the vacuum structure by a complimentary way-its physical respect. The key point is that MIT solution has an infinite action, which means there is no tunnelling between the non-integer topological field configrations although the continuous fractional gauge transformations which is allowed in Minkowski space eliminate the topological distinction between different winding number sections. Using this fact,we prove that the θ is zero at a semiclassical level.

In the classical picture, the vacuum of the non-abelian gauge theories is a many-state θ vacuum[1]. Only the integer winding number gauge transformations are allowed[2]. Correspondingly the vacuum states are the integer topological number field configurations which are connected by the instanton tunnelling[3]. This leads to the periodic picture in the one instanton sector. This periodicity contains an intrinsic θ angle which is similiar to the bloch wave number in the periodic potential case. However, all of this is in Euclidean space. Recently, MIT group[4] find that in Minkowski space, the continuous fractional gauge transformations are allowed. This leads to the collapse of the periodic picture. V.V.Khoze[5] showed that there is a unique vacuum actually, and the θ angle is zero.

In this paper, we look at this in a different way. We treat the vacuum as a periodic potential quantum problem and try to get the relation between the bloch wave number and the θ angle. Then we consider the solution [4] 's effect on this picture. We prove that the θ angle is zero only if the infinite action of the solution means the tunneling between the fractional winding number configurations are zero.

Since BPST[6] found the instanton solutions in Euclidean space, the vacuum structure is considered as the periodic θ Vacuum: Only the integer winding number field configurations are allowed in the vacuum, and the instanton is the most effective tunnelling path. The tunneling probability is really small[7] due to the topological suppress. Correspondingly the θ angle is very small. To look at the origin of the θ, we briefly review the vacuum picture: In the non-abelian gauge field theory:

$$S=-\frac{1}{2g^2}\int d^4xtr(F^{\mu\nu}F_{\mu\nu})$$

where the field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu]$, and $A_\mu=A_\mu^a\frac{\tau_a}{2}$, where τ_a,$a=1,2,3$ are the pauli matrix. As far as the topological properties of the vacuum structure are concerned, the $SU(2)$ theory is enough. g is the coupling constant.

First of all,in Euclidean space, we have many vacuum states[8]: $|n>$, $n=0$, ±1,±2,....$\pm\infty$, where n is the winding number of the gauge transformatioms U_n. So the vacuum

$$A_\mu^n=U_n(x)\partial_\mu U_n^{-1}(x)$$

is classified by the Pontriyagin index $\nu[A]$

$$\nu[A]=\frac{1}{16\pi^2}\int d^4xtr(\tilde{F}_{\mu\nu}F^{\mu\nu}) \tag{1}$$

where $\tilde{F}_{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual fields in Euclidean space. The equation (1) can be reduced to a surface integral which equals the change of the winding number in the boundary. In the vacumm section $\nu[A]$ is the winding number of the corresponding gauge transformations. The winding number $\mu[U]$ is defined as:

$$\mu[U]=-\frac{1}{24\pi^2}\int_{-\infty}^{\infty}d^3x\epsilon^{ijk}((U^+\partial_iU)(U^+\partial_jU(U^+\partial_kU)) \tag{2}$$

and $\mu[U_n] = n$. Actually the integrand of (2) is the jocobi matrix of the homotopy mapping from the group manifold to the Euclidean space boundary S^3:

$$S^3_G \longrightarrow S^3_E$$

If we demand any gauge transformations satisfy

$$U(x) \longrightarrow 1 \quad as \quad |x| \longrightarrow \infty$$

where $|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$. then the Euclidean space can be compactified as S^3 which leads to the above classification. So only integer winding numbers are allowed in this picture. And it is easy to show $U_n U_m = U_{m+n}$. Then in the Hilbert space:

$$|n + m >= g^{U_m}|n >$$

where g^{U_n} is the gauge transformation operator.

$$A_i^U = g^U A_i g^{U^+} = U^+(\partial_i + A_i)U$$

In the quantization of the theory, g^U is constructed by the field operators. However, it is required that the physical vacuum is a gauge invariant object. We need[3]:

$$g^U|vac >_{phys}= e^{-i\mu[U]\theta}|vac >_{phys} \tag{3}$$

the explicit construction of $|vac >_{phys}$ is

$$\begin{aligned} |vac >_{phys} &= |\theta >_{vac} \\ |\theta >_{phys} &= \sum e^{in\theta}|n > \end{aligned} \tag{4}$$

Here the θ angle is an arbitrary phase angle which can't be determined by any physical principle.

But the physical space time should be the Minkowski space time. In this case, the MIT solution serve the role as the instanton in Euclidean space. This solutions have the vacuum which contains continuous fractional winding number configurations instead of the dicrete integer field. From the equation (2), we know the gauge transformations are also continuously fractional. From the point of the homotopy, this implies the different winding number mapping U_n can be continuously deformed into each other now. So all the mapping $\{U_n\}$ are in the same homotopy class by the definition of homotopy. Therefore, there is no topological distinctions between the mappings. From the equation(2), we know all the vacuum states are in the same class. The vacuum state is unique from the point of topology. There is no periodic picture in the Minkowski space, then the θ does not exist. In other words, it is zero. This is the basic argument of the paper[5].

However, above is more mathematical way than physical picture.. We look at this at a different and more physical way. From the classical picture in the above,, we treat the many vacuum states as a periodic potential systems in the winding number space. That is $V(q) = V(q+n)$ with q as the coordinate of winding number. $V(q)$ is the potential of the gauge field, and n is a integer. We know that for a periodic potential, the wave function takes the bloch wave forms[9]:

$$\begin{aligned} |\psi> &= e^{ikq}U(q) \\ U(q) &= U(q+n) \end{aligned} \tag{5}$$

$U(q)$ is a periodic function in the winding number space. We construct it as

$$U(q) = \sum U(q+n)$$

Through some algebraic work, we have

$$|\psi> = \sum e^{-ikn'}U'(n') \tag{6}$$

where

$$U'(n') = e^{ikn'}U(n')$$

where $n' = q+n$.Here we ignore a total phase term. From quantum mechanics, we know that when the potential is very steep with minimum at integers, then the function is peaked at the minimum points(like $e^{-\alpha q^2}$). We can approximately treat the q' as zero which is the first minimum point.Now we have

$$|\psi> = \sum e^{-ikn}|n>$$

Compared with the equation(3), we can identify k as the θ. From here, we can see that the θ is the bloch wave number in the winding number space. In the following example, we can find that the θ is actually mixing angle due to the tunelling between different field configuration.

We proceed the bloch method furtherly by considering a periodic delta potential which is very close the vacuum picture in the Minkowski space in our approximation. Actually one can paramatrize the potential and mass as functions of winding number q[9] by using the exact tunneling path solution. Here the detail function is not important. Now we have the potential V(q)

$$V(q) = \lambda\delta(q-n)$$

where n is an arbitrary integer. the schrodinger equation is

$$\frac{d^2\psi}{dq^2} + \frac{2m}{\bar{h}^2}(E - V(q))\psi = 0$$

from bloch theorem, the solution of this equation will have the following properties:

$$\psi(q+n) = e^{i\theta n}\psi(q)$$

in the period $0 < q < 1$, the $\psi(q)$ has the form

$$\psi(q) = Ae^{i\beta q} + Be^{-i\beta q}$$

where $\beta = \sqrt{\frac{2mE}{\hbar^2}}$. According to the bloch theorem, the wave function in the period $1 < q < 2$ is

$$\psi(q) = e^{i\theta}(Ae^{i\beta(q-1)} + Be^{-i\beta(q-1)})$$

By using the connection conditions, one can find the non-traivial solutions are obtained only if the determinant of the coefficients of A and B vanishes, which gives the conditions:

$$\cos\theta = cos\beta + p\frac{\sin\beta}{\beta} \tag{7}$$

where $p = \frac{m\lambda}{\hbar^2}$. From the equation(7), one can get the energy band spectrum of this approximation of the vacuum structure Here we are only interested in the case when $\lambda \longrightarrow \infty$. In this case, to keep the equation (7), we need

$$\sin\beta = 0$$

to have finite energy solutions. So

$$\cos\theta = \pm 1 \tag{8}$$

So $\theta = \pi n$ with n as arbitrary non-zero integers. For the $\beta = 0$, we directly substitute into the determinant to see that it keeps the original determinat of the coefficients zero. Then $\beta = 0$ is also the solution as the $\lambda \longrightarrow \infty$. We see the tunneling goes to zero as the barrier between two different regions goes to infinity, then the mixing $e^{i\theta}$ goes to unity which means there is no mixing.

In the Yang-Mills field theory case, we know the action of MIT solution[4] is infinite. Also, from the physical meaning of this solution, we can see this solution is the connection path in physical space time between $\nu = 0$ and $\nu = q$:

The solution is to describle the moving spherical energy shell's effect on a finite region $r < R$: initially there is nothing in the region $r < R$ which means

$$A_\mu = \frac{1}{g}U\partial_\mu U^+ = 0$$

From(2), this means $\nu = 0$ at the initial situation. The energy shell is moving toward this region from $t = -\infty$, then after a long time, the energy shell is reflected

back to infinite at $t = \infty$, however the final state of the region $r < R$ is changed to a different configuration with $\nu = q$, and

$$A_\mu = \frac{1}{g} U \partial_\mu U^+$$

where

$$U = exp(iq\Lambda(r)\frac{\hat{x}\dot{\tau}}{2})$$

q is an arbitrary fractional number, and the $\Lambda(r)$ is satisfying the boundary conditions

$$\Lambda(r)|_{r\to 0} \longrightarrow 0, \qquad \Lambda(r)|_{r\to\infty} \longrightarrow 2\pi$$

The winding number of this U(r) is

$$\mu[U] = q - \frac{\sin(2\pi q)}{2\pi} \tag{9}$$

by calculating (2).

From this picture we see clearly the solution is a quantum path which connects the $\nu = 0$ with $\nu = q$.

Nontheless,the action is infinite which actually means there is no connection in the physical space time. Our key argument is as follows: In the nearby of this solutin, there are many other quantum path which also have infinite or very large action. And the transtion probability is proportional to e^{iS} where the S is the action of the path. So the large action means the very fast oscillation of the probability which eventually is washed out by the fast oscillation. Then actually there is no transition between these field configurations. Here we use the infinite potential barrier between the $\nu = 0$ and $\nu = q$ to realize this situation in the winding number space. In the above, we take the delta function with the integral height going to infinte to simulate the real case. However the distance between two potential peaks is not the unity but arbtrarily short due to the continuous q allowed. To consider this situstion, we change the the potential to $V(q) = \lambda\delta(q - na)$ with a goes to zero. In this potential, the equation is changed into:

$$\cos\theta = \cos\beta a + p\frac{\sin\beta a}{a\beta} \tag{10}$$

So when p goes to infinity, we get

$$\beta a = \pi n$$

with $n = 0, \pm 1, \pm 2, ...$ Then we take the limit $a \longrightarrow 0$ and keep the finite energy solutions, one immediately get

$$\beta a = 0$$

only the $n = 0$ is allowed. So we get

$$\cos\theta = \pm 1 \tag{11}$$

So only the $\theta = 0$ is physically reasonable.

Here, this conclusion is only depending on the assumption: periodic potential in the winding number space and the infinite energy barrier between fractional winding number field configurations in the Minkowski space. We should note this conclusion does not depend on the details of the periodic potentials, for example the periodic block potential, the periodic harminic potential. However, we assume the MIT solution is the most effective path in the above. This point is a still open problem[10]. Right now we don't know whether there is other finite action path or not. To proceed this respect, we need to solve the non-linear field eqautions both in the dual equation case[10] and the second order differential equation situation[5]. On the other hand, from above we know the θ angle is from the interference and tunnelling between different sector this is responsible for the smallness of the θ angle.

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